The Chaotic Nature of Three-dimensional Magnetic Topology Revealed by Transversely Intersecting Invariant Manifolds

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Introduction

Although adopted by Grad-Shafranov equation, EFIT, VMEC, *etc.*, the nested closed flux surface assumption does not necessarily hold when the axial symmetry of magnetic field is absent. An abundant amount of research has focused on how to stimulate a chaotic field layer at the plasma boundary to mitigate destructive type-I edge localized modes [1-5]. The transverse intersection of invariant manifolds is a signature of chaos, indicating the intrinsic unpredictability (of field line tracing) in the long run. Based on the theory of dynamical system and chaos [6-10], we offered the *invariant manifold growth formula in cyclindrical coordinates*, and revealed how the *X/O cycles shift under perturbation.* The symbol B is used to denote a general 3D vector field in our work, which is not required to be a magnetic field. Mathematicians might expect to try them on Lorenz/Rössler attractors.

$\mathcal{DP}^m$ evolution along cycles

The Jacobian of Poincare map, $\mathcal{DP}^m$, is acquired by a $2m\pi$ ϕ_e-integration of

$$\frac{\partial}{\partial \phi_e} \mathcal{D} X_{pol}(\phi_s, \phi_e) = \underbrace{\frac{\partial R B_{pol}/B_\phi}{\partial (R,Z)}(\phi_e)}_{:=\mathbf{A}(\phi_e)} \; \mathcal{D} X_{pol}(\phi_s, \phi_e), \tag{1}$$

where $X_{pol}(\phi_s, \phi_e, x_0)$ is a field line initiating from $x_0 = (x_{0R}, x_{0Z})^T$ at ϕ_s. $\mathcal{D}$ denotes the derivative *w.r.t.* x_0. Actually, $\mathcal{D} X_{pol}(\phi_s, \phi_e)$ is the state transition matrix of $x' = \mathbf{A} x$. It is natural to be curious on the relationship of $\mathcal{DP}^m(\phi)$ at different ϕ of a cycle so that one can save time by avoiding repeated integration from ϕ_1 to $\phi_1 + 2m\pi$, from ϕ_2 to $\phi_2 + 2m\pi$, ……We deduced this relationship after tedious operations. Now we know

$\mathcal{DP}^m(\phi)$ on a cycle satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\mathcal{DP}^{\pm m}(\phi) = \left[\mathbf{A}(\phi),\ \mathcal{DP}^{\pm m}(\phi)\right], \tag{2}$$

which is named the $\mathcal{DP}^m$ *evolution* formula. Obviously this is a system of ordinary differential equations of the 2×2 matrix $\mathcal{DP}^m(\phi)$, the eigenvectors of which determine from which the invariant manifolds grow.

Invariant manifold growth formula

By tracing from all points on the cycle and analyzing the relevant differentials (see Fig. 1), the classical simple field line tracing equation is extended to the *invariant manifold growth* formula in cylindrical coordinates,

$$\frac{\partial X^{u/s}}{\partial s} = \frac{\dfrac{RB_{pol}}{B_\phi} - \dfrac{\partial X^{u/s}}{\partial \phi}}{\pm\left\|\dfrac{RB_{pol}}{B_\phi} - \dfrac{\partial X^{u/s}}{\partial \phi}\right\|_2}, \tag{3}$$

the initial condition of which is that $\partial_s X^{u/s}(s,\phi)|_s = 0$ equals the normalized eigenvector of $\mathcal{DP}^m(\phi)$. The two parameters of the manifold are the azimuthal angle ϕ and the curve (intersected by the invariant manifold and the R-Z section at ϕ angle) length s in the R-Z section, as shown below.

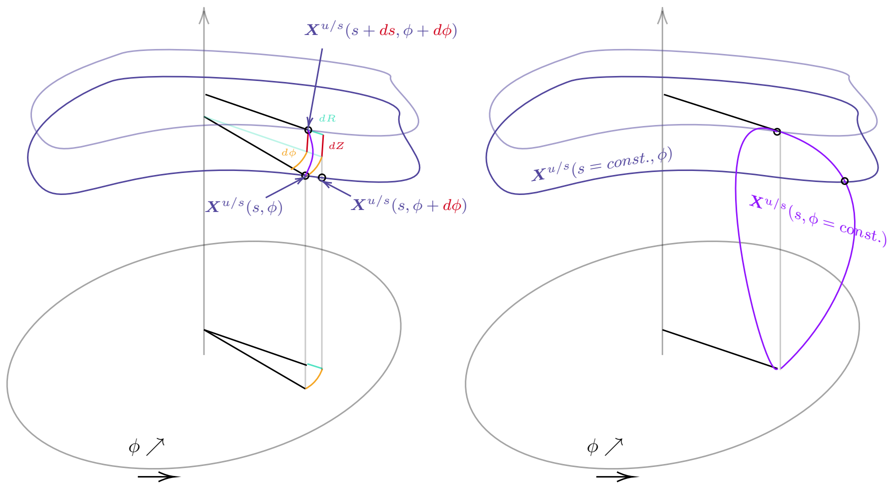


Figure 1: Geometric diagram to show the relationship among the differentials $\mathrm{d}s$, $\mathrm{d}\phi$, $\mathrm{d}R$ and $\mathrm{d}Z$, which are used in the deduction of the invariant manifold growth formula. It is supposed that there exists a limit cycle at bottom, from which an invariant manifold grows.

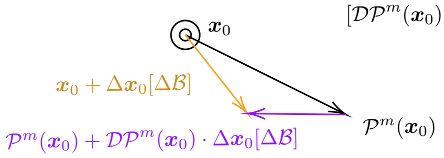
Orbit and cycle shift

We further regard the whole magnetic field as a functional argument of Poincaré map and utilize the functional (Fréchet) derivative from functional analysis (see [11] for a concise introduction) to obtain the following results concerning the orbit/cycle behaviour under perturbation. Before we study X/O cycles, we need to begin with the general orbits in a vector field. The following orbit shift formula under perturbation depicts the shift of an orbit $X_{pol}[\mathcal{B}](\phi)$ under the perturbation $\Delta\mathcal{B}$, where the purple term $q(\phi)$ distinguishes the equation from the similar differential equations of $\mathcal{D}X_{pol}$. $\delta X_{pol}[\mathcal{B};\Delta\mathcal{B}](\phi)$ is the first variation of $X_{pol}[\mathcal{B}](\phi)$ *w.r.t.* $\Delta\mathcal{B}$.

$$\frac{\partial}{\partial\phi}\delta X_{pol}[\mathcal{B};\Delta\mathcal{B}](\phi) = \overbrace{\frac{\partial(RB_{pol}/B_\phi)}{\partial(R,Z)}(\phi)}^{:=\mathbf{A}(\phi)}\ \delta X_{pol}[\mathcal{B};\Delta\mathcal{B}](\phi) + \overbrace{\begin{bmatrix} R/B_\phi & 0 & -RB_R/B_\phi^2 \\ 0 & R/B_\phi & -RB_Z/B_\phi^2 \end{bmatrix}(\phi) \underbrace{\left(\frac{\delta\left[B_R,B_Z,B_\phi\right]^T}{\delta\mathcal{B}}\Delta\mathcal{B}\right)}_{\text{simply } =\left[\Delta B_R,\Delta B_Z,\Delta B_\phi\right]^T}}^{:=q(\phi)} \tag{4}$$

A $2m\pi$ ϕ-integration of the equation above tells in which direction the Poincare map $\mathcal{P}^m(x_0)$ is pushed away from x_0, and how far. Notice this equation is a linear inhomogeneous one with zero initial condition $\delta X_{pol}[\mathcal{B};\Delta\mathcal{B}](0) = 0$, hence the solutions for different $\Delta\mathcal{B}$ can be superposed linearly.

As one can easily imagine, both the original x_0 and $\mathcal{P}^m[\mathcal{B}+\Delta\mathcal{B}](x_0)$ are no longer on the cycle. To locate the new x0 point on the cycle, we utilize DPm(x0) as follows,



$$\begin{aligned} \boldsymbol{x}_0+\Delta\boldsymbol{x}_0 &= \mathcal{P}^m(\boldsymbol{x}_0)+\mathcal{DP}^m(\boldsymbol{x}_0)\cdot\Delta\boldsymbol{x}_0 \\ [\mathcal{DP}^m(\boldsymbol{x}_0)-I]\cdot\Delta\boldsymbol{x}_0 &= \boldsymbol{x}_0-\mathcal{P}^m(\boldsymbol{x}_0) \\ \Delta\boldsymbol{x}_0[\Delta\mathcal{B}] &= [\mathcal{DP}^m(\boldsymbol{x}_0)-I]^{-1}\cdot \underbrace{[\boldsymbol{x}_0-\mathcal{P}^m(\boldsymbol{x}_0)]}_{=-\delta\boldsymbol{X}_{pol}[\mathcal{B};\Delta\mathcal{B}](2m\pi)+o(\Delta\mathcal{B})} \end{aligned}$$

This is *X/O-point shift formula under perturbation*. Undoubtedly, the most important perturbation field is the derivative of magnetic field itself *w.r.t.* time, *i.e.* $\partial\mathcal{B}/\partial t$. If $\Delta\mathcal{B}$ is substituted for $\partial\mathcal{B}/\partial t$, the formula will give the X/O point shift velocity. The factor $[\mathcal{DP}^m - \mathbf{I}]^{-1}$ decides that the formula does not work for cycles on the rational flux surfaces of magnetic fields, because their $\mathcal{DP}^m$ have eigenvalues 1, which make it impossible to do the matrix invert.

Since we have known how to calculate $\Delta x_0[\Delta\mathcal{B}]$ at a single R-Z section, one might be curious on how $\Delta x_0[\Delta\mathcal{B}](\phi)$ at different ϕ are related, like $\mathcal{DP}^m$ *evolution* formula. The derivative of $\Delta x_0[\Delta\mathcal{B}](\phi)$ *w.r.t.* ϕ turns out to be (also after tedious computation)

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\Delta x_0 = \mathbf{A}\Delta x_0 + q(\phi) \tag{5}$$

$$\text{or more strictly} \qquad \frac{\partial}{\partial\phi}\delta\Delta x_0[\boldsymbol{0};\Delta\mathcal{B}](\phi) = \mathbf{A}(\phi)\ \delta\Delta x_0[\boldsymbol{0};\Delta\mathcal{B}](\phi) + q(\phi) \tag{6}$$

Now we can discuss about *X/O-cycle* shift formula instead of just *X/O-point*.

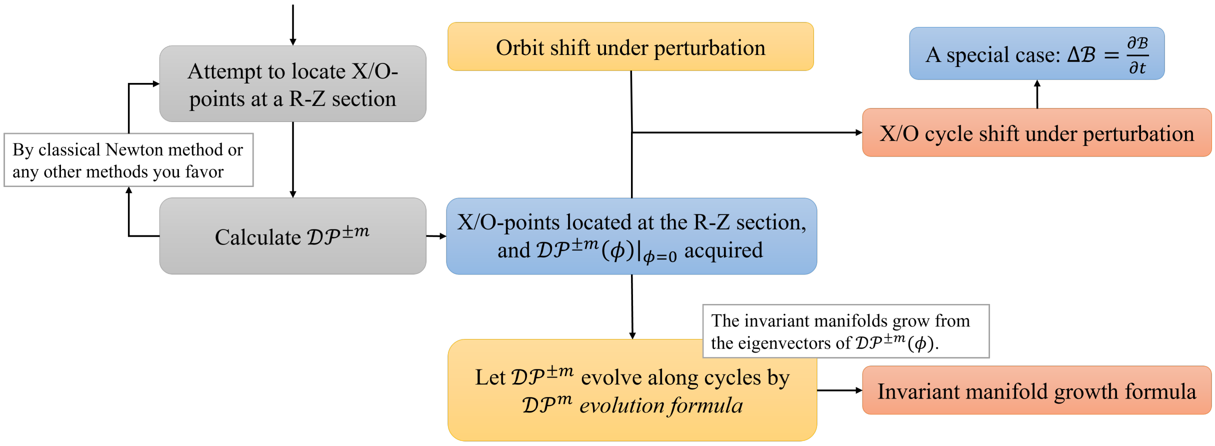


Figure 2: Thought process mapping of the formulas given in our work

Conclusion and discussion

Our existing results concerning the invariant manifolds of a general 3D vector field are concluded in Fig. 2. Numeric implementation is planned to keep up the progress of analysis.

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