On Relativistic Braginskii Transport Equations: Mixed Approach

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1 Introduction

- In recent decades it has been shown [1,2] that the relativistic effects in fusion plasmas with temperatures of tens of keV, despite $T_e \ll m_e c^2$, can produce non-negligible effects in transport. Still, nearly all predictions for fusion reactor scenarios, even for aneutron projects with $T_e \sim 100$ keV, have been made using nonrelativistic transport theory [3].
- In the present work, we consider a hot plasmas with fully relativistic electrons and assume that the hydrodynamic flows are slow compared to the thermal velocity. Such consideration, fully relativistic for thermal plasma and weakly relativistic for fluxes, we call "mixed approach".
- In addition, improvements in the methods for solving the relativistic kinetic equation are proposed. Since the main aim of this work is to include the relativistic effects and to keep the classic-like form of the equations, we do not use 4-vectors formalism.

1. I. Marushchenko *et al.*, PPCF, 55, 085005 (2013); 2. G. Kapper *et al.*, Phys. Plasmas, 25, 122509 (2018); 3. M. Kikuchi, K. Lackner, M.Q. Tran, Fusion Physics, IAEA, Vienna (2012).

2 Derivation of the relativistic Braginskii equations

- Relativistic kinetic equation (RKE) for electrons in divergent form:

$$\frac{\partial f_e}{\partial t} + \frac{\partial}{\partial x_k}(v_k f_e) + \frac{\partial}{\partial u_k}\left(\frac{e}{m_e}\big(E_k + \frac{1}{c}[\mathbf{v}\times\mathbf{B}]_k\big) f_e\right) = C_{ee}(f_e) + C_{ei}(f_e), \tag{1}$$

where $u_k = v_k\gamma$, $\gamma = \sqrt{1+u^2/c^2}$.

- We assume that electrons are near the thermodynamic equilibrium, given by drifting Maxwell-Jüttner distribution function with drift velocity $\mathbf{V}$:

$$f_{e0} = C_{MJ}\frac{n_e}{\pi^{3/2}u_{te}^3}\exp\left[-\mu\gamma_0\Big(\gamma - \frac{1}{\gamma_0} - \frac{V_k u_k}{c^2}\Big)\right], \tag{2}$$

with $\mu = m_e c^2/T_e \gg 1$; the flow factor γ_0 is taken in weakly relativistic limit, $\gamma_0 \simeq 1 + V^2/2c^2$; the normalizing coefficient equals

$$C_{MJ} = \sqrt{\frac{\pi}{2\mu}}\frac{\mathrm{e}^{-\mu}}{K_2(\mu)} = 1 - \frac{15}{8\mu} + O\Big(\frac{1}{\mu^2}\Big), \tag{3}$$

with $K_n(x)$ as the modified Bessel function of second kind of the n-th order.

- Relation between the thermal energy and temperature can be represented as

$$W \equiv n_e m_e c^2\langle\gamma' - 1\rangle = \Big(\frac{3}{2} + \mathcal{R}\Big)n_e T_e \quad \text{with} \quad \mathcal{R} = \mu\Big(\frac{K_3(\mu)}{K_2(\mu)} - 1\Big) - \frac{5}{2} = \frac{15}{8\mu} + O\Big(\frac{1}{\mu^2}\Big). \tag{4}$$

Similarly, one can define the heat flux, which for the rest frame is coupled with momentum (which is a purely relativistic effect),

$$q_k = n_e m_e c^2\langle(\gamma' - 1)v_k'\rangle \quad \text{and} \quad n_e m_e\langle u_k'\rangle = \frac{1}{c^2}q_k. \tag{5}$$

The momentum flux, $n_e m_e\langle v_k' u_j'\rangle = p_e\delta_{kj} + \pi_{kj}$, is decomposed in two parts, the hydrostatic scalar pressure, p_e, and the (traceless) stress viscous tensor π_{kj},

$$p_e = \frac{1}{3}n_e m_e\Big\langle\frac{u'^2}{\gamma'}\Big\rangle = n_e T_e, \quad \text{and} \quad \pi_{kj} = n_e m_e\langle v_k' u_j'\rangle - p_e\delta_{kj}. \tag{6}$$

- The moments related to the collisional operator are also required. Electron-ion collisional friction force, R_k^{ei},

$$R_k^{ei} = \int m_e u_k' C_{ei}' d^3u'. \tag{7}$$

- The rate of collisional energy exchange between relativistic electrons and classical ions can be written as follows:

$$P^{ei} = \int m_e c^2(\gamma' - 1)C_{ei}' d^3u' = P_{(cl)}^{ei}C_{MJ}(\mu)\Big(1 + \frac{2}{\mu} + \frac{2}{\mu^2}\Big), \tag{8}$$

where $P_{(cl)}^{ei}$ is classical (non-relativistic) rate of e-i energy exchange, $P_{(cl)}^{ei} \propto -\frac{T_e - T_i}{T_e^{3/2}}$.

- Lorentz transformation to the rest frame in weakly relativistic limit:

$$u_k \simeq u_k' + \gamma_0\gamma' V_k + \frac{V_k V_j}{2c^2}u_j', \qquad \gamma \simeq \gamma_0\gamma' + \frac{V_j u_j'}{c^2}, \tag{9}$$

with infinitesimal volume conservation, $d^3u/\gamma = d^3u'/\gamma'$.

- Direct integration of Eq. (1) leads to continuity equation for density,

$$\frac{\partial}{\partial t}(\gamma_0 n_e) + \frac{\partial}{\partial x_k}(\gamma_0\Gamma_k) = 0. \tag{10}$$

Formally, this equation has exactly the same form as in the fully relativistic approach. The weakly relativistic expansion for γ_0 is assumed, but not shown.

- Then, integrating Eq. (1) with the weight $n_e m_e u_k$, we get the momentum balance equation,

$$\frac{\partial}{\partial t}\Big[n_e m_e\big(V_k + \delta U_k\big)\Big] + \frac{\partial}{\partial x_j}\Big(\Pi_{kj} + \delta\Pi_{kj}\Big) = en_e E_k + \frac{1}{c}\big[\mathbf{J}\times\mathbf{B}\big]_k + \big(R_k^{ei} + \delta R_k^{ei}\big). \tag{11}$$

- Here and below, the relativistic corrections, which disappear with $V/c \to 0$, are labeled by δ.

- The correction for the momentum is

$$\delta U_k \simeq \frac{1}{nmc^2}\Big(q_k + \big(W + p\big)V_k\Big) + \frac{1}{c^2}\pi_{kj}V_j; \tag{12}$$

the stress tensor is classical-like, $\Pi_{kj} = p_e\delta_{kj} + \pi_{kj} + n_e m_e V_k V_j$, while correction is

$$\delta\Pi_{kj} \simeq \frac{V_k V_j}{c^2}\Big(W + p\Big) + \frac{1}{c^2}(q_j V_k + q_k V_j) + \frac{V_s}{2c^2}(\pi_{ks}V_j + \pi_{js}V_k). \tag{13}$$

Correction for friction force (see also Eq. (7)) is

$$\delta R_k^{ei} \simeq \frac{V_k}{c^2}\Big(P^{ei} + \frac{1}{2}V_j R_j^{ei}\Big). \tag{14}$$

The electric current is $\mathbf{J} = en_e\mathbf{V} = en_e(\mathbf{V}_e - \mathbf{V}_i)$.
Integrating Eq. (1) with the weight $n_e m_e c^2(\gamma - 1)$, we get the energy balance equation,

$$\frac{\partial}{\partial t}\Big(W + K + \delta\mathcal{E}\Big) + \frac{\partial}{\partial x_k}\big(Q_k + \delta Q_k\big) = J_k E_k + R_k^{ei}V_k + \big(P^{ei} + \delta P^{ei}\big). \tag{15}$$

Here, W is defined by Eq. (4), $K = nmV^2/2$ and relativistic correction is

$$\delta\mathcal{E} = \frac{V^2}{c^2}\Big(W + p\Big) + \frac{1}{c^2}(\pi_{ij}V_i + q_j)V_j. \tag{16}$$

The energy flux is also classical-like,

$$Q_k = \Big(W + p + K\Big)V_k + q_k + \pi_{kj}V_j, \tag{17}$$

while its relativistic correction is

$$\delta Q_k = \frac{V^2}{c^2}\Big(W + \frac{p}{2}\Big)V_k + \frac{V^2}{2c^2}\big(\delta_{jk} + \frac{3V_jV_k}{V^2}\big)q_j + \frac{1}{2c^2}\Big(\pi_{ij}V_iV_j\Big)V_k. \tag{18}$$

The correction for the rate of collisional e-i energy exchange is equal to

$$\delta P^{ei} = \frac{V^2}{2c^2}P^{ei}. \tag{19}$$

Eq. (10), Eq. (11) and Eq. (15) form a set of relativistic transport equations.

3 On solving linearized relativistic kinetic equations

- Sonine polynomials are the best only for classical limit.
- Proposed: generalized Laguerre polynomials $L_n^{(\alpha)}(\kappa)$ of order $\alpha = 3/2 + \mathcal{R}(\mu)$, where $\mathcal{R} = 15/8\mu + ...$ and $\kappa = \mu(\gamma - 1)$. Advantage: they are eigenfunctions for the RHS of linearized RKE for arbitrary T_e.
- Let us demonstrate the method for $V = 0$. In this case, linearized relativistic kinetic equation is:

$$\frac{e}{m_e c}[\mathbf{v}\times\mathbf{B}]_k\frac{\partial f_{e1}}{\partial u_k} - C_e^{lin}(f_{e1}) = -v_k\Big(A_k^{(1)} + \Big(\kappa - \frac{5}{2} - \mathcal{R}\Big)A_k^{(2)}\Big)f_{e0}, \tag{20}$$

where $A_{1,2}$ are the thermodynamic forces,

$$\mathbf{A}^{(1)} = \nabla\log p_e + \frac{e\mathbf{E}}{T_e} \quad \text{and} \quad \mathbf{A}^{(2)} = \nabla\log T_e. \tag{21}$$

- The angle-dependence: spherical Legendre harmonics.
- The energy dependence: $L_n^{(\alpha)}(\kappa)$. Since $L_0^{(\alpha)}(\kappa) = 1$ and $L_1^{(\alpha)}(\kappa) = \frac{5}{2} - \kappa - \mathcal{R}$, RHS of Eq. (20), contains linear combination of only two terms.
- Series converges as in classical limit and remains valid for arbitrary T_e.
- It can be shown that all transport coefficients are expressed by the integrals as follows,

$$\begin{aligned} M_{nm}^{ab} &= \tfrac{\tau_{ab}}{n_a}C_{MJ}^{-1}\int \mathrm{d}^3u\, u_k L_n^{(\alpha)}(\kappa)C^{ab}\Big(\tfrac{m_a u_k}{T_a}L_m^{(\alpha)}(\kappa)wf_{a0}; f_{b0}\Big),\\ N_{nm}^{aa} &= \tfrac{\tau_{aa}}{n_a}C_{MJ}^{-1}\int \mathrm{d}^3u\, u_k L_n^{(\alpha)}(\kappa)C^{aa}\Big(f_{a0}; \tfrac{m_a u_k}{T_a}L_m^{(\alpha)}(\kappa)wf_{a0}\Big), \end{aligned} \tag{22}$$

where $w = \kappa^{\mathcal{R}}\Big(\frac{2}{\gamma+1}\Big)$.

- These coefficients have the same structure as in classical limit.

Summary 1: Braginskii equations are derived in mixed approach, with fully relativistic description for thermal electrons and weakly relativistic for hydrodynamic electron flow.

Summary 2: Instead of Sonine polynomials, it is proposed to solve the linearized RKE by expansion in generalised Laguerre polynomials $L_n^{(3/2+\mathcal{R})}(\kappa)$, which are eigenfunctions of RHS of relativistic kinetic equation.