Quantum Computing Approach to Electromagnetic Wave Propagation in Plasma

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Introduction

Quantum computing (QC) has shown enormous promise for solving classes of problems for which a quantum algorithm can obtain a speedup (advantage) over the classical counterpart. Such a speed up would be extremely desirable to be exploited in favor of classical,high demanding in computational resources, simulations of electromagnetic wave propagation in plasma.

We propose a formalism, appropriate for application of Quantum Computation methods in plasma electromagnetic wave propagation, based on the Schrodinger representation of Maxwell Equations for a cold, homogeneous, collision-less, two species magnetized plasma.

The "quantal" electromagnetic picture retains all the attributes of classical theory whereas provides new insights in the framework of plasma electrodynamics as well as to other quantum-based descriptions.

Keywords: Schrodinger Equation, Plasma Electrodynamics, Quantum Computing, Maxwell Equations.

Quantum Computation

In QC, the fundamental carrier of information is named quantum bit or qubit and it is represented as vector state in Hilbert space $\mathscr{H}_2$ that reflects a two-level quantum system.

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle\,, \quad |a|^2 + |b|^2 = 1 \tag{1}$$

The prowess of QC can be summarized in the following qubit properties [1].

• **Propabilistic nature**: Performing a measurement with the projective operator $P_0 = |0\rangle\langle 0|$ in qubit (1) we obtain

$$|\psi\rangle' = \frac{a\,|0\rangle}{|a|}, \quad p(0) = |a|^2$$

• **Superposition**: If we allow n qubits to interact the combined Hilbert space is produced through tensor product $\mathscr{H} = \otimes^{\times n}\mathscr{H}_2$. As a result, the computational basis set in the combined space consists of 2^n elements.

$$\mathscr{H} \ni |\psi\rangle = \sum_{j=0}^{2^n-1} c_j |j\rangle$$

• **Entanglement**: Hilbert space $\mathscr{H}$ contains non-separable states.

$$\exists |\psi\rangle \in \mathscr{H} \Rightarrow |\psi\rangle \neq \bigotimes_{i=1}^{n} |\psi_i\rangle, \forall |\psi_i\rangle \in \mathscr{H}_2$$

Finally, the evolution of quantum system is provided by Schrodinger equation and specifically by the Hermitian generator of dynamics $\hat{H} = \hat{H}^\dagger$ which corresponds to the energy operator.

$$i\frac{\partial |\psi\rangle}{\partial t} = \hat{H}|\psi\rangle, \quad |\psi\rangle = \hat{\mathscr{U}}|\psi_0\rangle = \exp\{-it\hat{H}\}|\psi_0\rangle \tag{2}$$

Simulation of quantum evolution (2) for $\hat{H} = \sum_{\gamma=1}^{\Gamma} H_\gamma$ can be accomplished incorporating p-order product formulas $S(t)$. Assuming, time step t/r we want to construct unitary gates $S(t/r)$ for a quantum simulation with Trotter error

$$\left\| S^r(t/r) - e^{it\hat{H}} \right\| = \mathscr{O}\left(\frac{(\sum_{\gamma=1}^{\Gamma} \|H_\gamma\| t)^{p+1}}{r^p} \right) \tag{3}$$

Maxwell Equations for Plasma

Maxwell equations in temporal domain for cold, homogeneous, magnetized plasma, consisted of electrons and ions read

$$i\frac{\partial u}{\partial t} = \hat{D}u = \left[W_0^{-1}\hat{M} - iW_0^{-1}\frac{\partial G(t)}{\partial t} * \right] u = W_0^{-1}\hat{M}u - iW_0^{-1}\int_0^t \frac{\partial G(t-\tau)}{\partial t} u(r,\tau) d\tau \tag{4}$$

the Maxwell operator $\hat{M}$ and matrices W_0, G have the form

$$\hat{M} = i\begin{bmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{bmatrix}, \quad W_0 = \begin{bmatrix} \varepsilon_0 I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & \mu_0 I_{3\times3} \end{bmatrix}, \quad W_0{}^{-1}G = \begin{bmatrix} K(t-\tau) & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}$$

The electric susceptibility kernel $K(t)$ is given as

$$K(t) = \sum_{n=e,i} \begin{bmatrix} \frac{\omega_{pn}^2}{\omega_{cn}} \sin\omega_{cn}t & -\frac{\omega_{pn}^2}{\omega_{cn}}\left(\cos\omega_{cn}t - 1\right) & 0 \\ \frac{\omega_{pn}^2}{\omega_{cn}}\left(\cos\omega_{cn}t - 1\right) & \frac{\omega_{pn}^2}{\omega_{cn}} \sin\omega_{cn}t & 0 \\ 0 & 0 & \omega_{pn}^2 t \end{bmatrix}$$

Maxwell equation (4) obeys the following physical postulates: *Determinism, Linearity, Causality, Locality in space* and *Time translation invariance*.

Whereas Maxwell equation (4) has the same form with Schrodinger equation (2) it possess significant differences:**The state vector $u = (E,H)^T$ is real, and the generator of dynamics $\hat{D}$ does not corresponds to energy operator and it is not a Hermitian.**

In order to obtain a conservative system, we extend the electromagnetic system considering the polarization as an independent entity of electric field E. Such an extension is possible due to the fact that plasma is a passive medium [2](energy density $e(r,t) \geq e(r,0)$).

$$\frac{\partial(\omega\tilde{K}(\omega))}{\partial\omega} > 0\;(positive\,definite\,matrix)$$

Schrodinger Equation for Plasma

The Hermitian representation of (3) in the extended system with auxiliary fields ϕ is

$$i\frac{\partial\psi(r,t)}{\partial t} = \begin{bmatrix} 0_{3\times3} & icS\cdot\hat{p} & -i\omega_{pe}A & -i\omega_{pi}A & -i\sqrt{\omega_{pe}^2+\omega_{pi}^2}C \\ -icS\cdot\hat{p} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ i\omega_{pe}A & 0_{3\times3} & i\omega_{ce}B & 0_{3\times3} & 0_{3\times3} \\ i\omega_{pi}A & 0_{3\times3} & 0_{3\times3} & i\omega_{ci}B & 0_{3\times3} \\ i\sqrt{\omega_{pe}^2+\omega_{pi}^2}C & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \end{bmatrix}\psi(r,t) \tag{5}$$

with definitions

$$\psi(r,t) = \begin{bmatrix} \sqrt{\varepsilon_0}E \\ \frac{B}{\sqrt{\mu_0}} \\ \sqrt{\varepsilon_0}\omega_{pe}\phi_e \\ \sqrt{\varepsilon_0}\omega_{pi}\phi_i \\ \sqrt{\varepsilon_0(\omega_{pe}^2+\omega_{pi}^2)}\phi_b \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S = (S_x, S_y, S_z), \quad S_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad S_y = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \quad S_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\phi_{e,i} = \int_0^t \begin{bmatrix} \cos\omega_{ce,i}(t-\tau) & \sin\omega_{ce,i}(t-\tau) & 0 \\ -\sin\omega_{ce,i}(t-\tau) & \cos\omega_{ce,i}(t-\tau) & 0 \\ 0 & 0 & 0 \end{bmatrix} E(r,\tau)d\tau, \quad \phi_b = \int_0^t CE(r,\tau)d\tau$$

Hermitian Schrodinger representation of the extended electromagnetic system (5) has been derived though a unitary mapping $\mathscr{W}^{1/2} : L^2_{\mathscr{W}}(\Omega) \to L^2(\Omega)$ due to the pseudo-Hermitian [3] nature $\langle\Phi,\Psi\rangle_{\mathscr{W}} = \langle\Phi,\mathscr{W}\Psi\rangle = \langle\mathscr{W}^{1/2}\Phi,\mathscr{W}^{1/2}\Psi\rangle = \langle\phi,\psi\rangle$ of the system.

$$i\frac{\partial\psi(r,t)}{\partial t} = \hat{D}(r)\psi(r,t), \quad \psi(r,0) = \psi_0, \quad \psi \in L^2(\Omega) \tag{6}$$

Most importantly, we can separate the action of Hermitian and traceless operator $\hat{D}$ in two distinct parts.

$$\hat{D}\psi(r,t) = \hat{D}_{vac}\psi(r,t) + \hat{D}_{disp}\psi(r,t) = \hat{D}_{vac}\psi_{vac}(r,t) + \hat{D}_{disp}\psi(r,t) \tag{7}$$

Conclusions

• We have constructed a Hermitian Schrodinger representation of Maxwell equations for plasma that admits unitary evolution. The inner product $\langle\psi|\psi\rangle$ denotes the generalized electromagnetic energy.

$$E = \frac{1}{2}\langle u, Wu\rangle + \int_0^t \int_0^s \int_\Omega u^T(r,s)\frac{\partial G(r,s,\tau)}{\partial s}u(r,\tau)d\,r d\,\tau d\,s$$

• In that way a generalized Poynting Theorem can be constructed with Dirichlet boundary conditions.

$$\frac{\partial e}{\partial t} + \nabla \cdot S = 0$$

• The RSW vector is a proper complexification only for vacuum and inhomogeneous dispersionless cases [4].

• The separation of generator of evolution $\hat{D} = \hat{D}_{vac} + \hat{D}_{disp}$ proposes a Trotterization procedure for simulation of evolution. In addition, the evolution operator belongs to $SU(15)$ an attribute that can improve the error (3) of Trotterization [5].

• A quantum simulation in a discetized 3D lattice with N nodes can be implemented with $\log_2 N + 4$ qubits.

$$|\tilde{\psi}\rangle = \sum_{i=0}^{14}\sum_{m=0}^{N-1} a_{i,m}\,|e_i\rangle \otimes |m\rangle$$

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