Dynamical study of chaotic magnetic fields in magnetohydrodynamic plasmas

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We consider nonlinear evolution of chaotic magnetic fields in flowing magnetohydrodynamic (MHD) plasmas. The evolution is described by a set of coupled nonlinear equations with the assumption that all quantities vary only in one direction. Chaotic oscillations of the magnetic fields have been analysed numerically in a general sense considering all parameters and oscillations of lower hybrid and cyclotron types are also seen for some specific cases. Fixed points accompanied with their stability analysis corresponding to the nonlinear evolution equations are performed. The behaviour of Lyapunov exponent and correlation dimension in parameter space are explored; in addition, long range correlations and anticorrelations of the oscillations using Hurst exponent estimated by the method of rescaled range (R/S) statistics have been analysed. Our results can have implications in laboratory as well as space and astrophysical plasmas.

Extensive investigations on chaotic magnetic fields have been witnessed recently in different systems with a predominance in plasmas [1–9] including different laboratory, space and astrophysical situations. This is because plasmas consist of different types of charged particles which require magnetic fields for their confinement. Generally chaotic magnetic fields in plasmas have been analyzed in the frameworks of fluid and kinetic models. The study of chaotic magnetic fields using magnetohydrodynamics (MHD) models in plasmas is very rare. This kind of work has been reported by Lee and Parks [1] three decades ago. In their work [1], Lee and Parks have derived a set of coupled nonlinear magnetic field equations in flowing MHD plasmas with the assumption that all quantities vary only in one direction. Then they have proved the chaotic nature of the nonlinear magnetic fields by casting their governing coupled equations in the form of forced Duffing equations which can involve chaotic dynamics [2]. As such, they have not done a detailed dynamical study of the chaotic magnetic fields in their system of flowing MHD plasmas using the original derived set of coupled nonlinear equations in the contexts of many important techniques of nonlinear analysis [10].

In this regard, a detailed nonlinear dynamical analysis of chaotic developments of nonlinear magnetic fields in flowing MHD plasmas using various techniques like lyapunov exponent (LE), phase space, bifurcations, fast Fourier transform (FFT) etc. [10] along with various other techniques of time series analysis deserve to be very interesting. This has been explored predominantly in the present work. In addition, several other tools such as correlation dimension (CD) [11], which can be used to distinguish between deterministic chaos and random noise, and Hurst exponent (HE) [12] have been employed in our work for investigating the fractal nature, in particular, long range behaviour of chaotic magnetic fields in our system. This is because CD can efficiently be regarded as a type of fractal dimension whereas HE is directly related to fractal dimension. The analysis based on various other types of fractal dimensions has been planned to be done in future. Some preliminary results based on the self-similar or fractal nature of chaotic magnetic fields in flowing MHD plasmas have been reported in this article.

We consider the same system of flowing MHD plasmas as analysed by Lee and Parks [1]. In their formulation [1] based on mixed two fluid MHD model, the magnetic field $\vec{B}$ is constant in $x-$direction, i.e. $B_x = B_{xo} = B_0$, and the y and z components of $\vec{B}$ are governed by the following normalized coupled nonlinear equations:

$$\frac{d^2B_y}{dt^2}+a\frac{dB_z}{dt}+b\frac{dB_y}{dt}-[(\alpha {M_A}^2-1)+(e^2+f^2)]B_y+{B_y}^3+B_y{B_z}^2-e({B_y}^2+{\beta_z}^2)+e[(\alpha {M_A}^2-1)+(e^2+f^2)]=0, \quad (1)$$

$$\frac{d^2B_z}{dt^2}-a\frac{dB_y}{dt}+b\frac{dB_z}{dt}-[(\alpha {M_A}^2-1)+(e^2+f^2)]B_z+{B_z}^3+B_z{B_y}^2-f({B_y}^2+{B_z}^2)+f[(\alpha {M_A}^2-1)+(e^2+f^2)]=0, \quad (2)$$

where the dimensionless parameters are given as: $a=\sqrt{\frac{\Omega_e}{\Omega_i}}-\sqrt{\frac{\Omega_i}{\Omega_e}}$, $b=\frac{\nu}{\alpha\sqrt{\Omega_e\Omega_i}}$, $\alpha=1+\frac{m_e}{m_i}$, $M_A=\frac{v_0}{v_A}$, $e=(B_y)_{t=0}$ and $f=(B_z)_{t=0}$. Here, Ω_e and Ω_i represent electron and ion cyclotron frequencies respectively; ν is electron-ion collision frequency; m_e and m_i denote the masses of electron and ion respectively; M_A is Alfven Mach number; v_0 is flow velocity of plasma; v_A is Alfven velocity given as: $v_A=\frac{B_0}{\sqrt{4\pi\rho_i}}$ where ρ_i represents ion charge density. The following normalizations have been used in our system: $B_y \longrightarrow \frac{B_y}{B_0}, B_z \longrightarrow \frac{B_z}{B_0}$ and $t \longrightarrow \sqrt{\Omega_e\Omega_i}t$.

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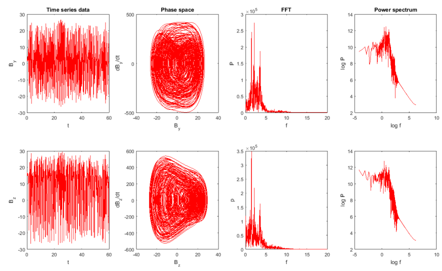

FIG. 1: Chaotic solutions of coupled nonlinear Eqs. (1) and (2) accompanied with plots of their corresponding phase space, FFT and Power spectrum for $a = 9$, $b = 0.005$, $\alpha = 1.032$, $M_A = 0.0001$, $e = 2$, $f = 15$, $(\frac{dB_y}{dt})_{t=0} = 10$ and $(\frac{dB_z}{dt})_{t=0} = 11$

The nature of magnetic field fluctuations governed by Eqs. (1) and (2) in the linear limit with the simplified assumption of vanishing initial conditions has been found to be of lower hybrid as well as cyclotron types under certain approximations. Similarly, oscillations of only lower hybrid types have been recovered if we further neglect electron-ion collisions along with the assumption of a two-dimensional magnetic field, i.e. either $B_y = 0$ or $B_z = 0$. We have explicitly mentioned these special cases in order to correlate magnetic field fluctuations in confrontation with fluctuations of the constituent charged particles in the plasma system considered. This is because oscillations of lower hybrid, upper hybrid and cyclotron types are very well-known to represent certain basic modes of oscillations corresponding to the constituent ion and electron species in magnetized plasmas. The interesting point in this context is that magnetic field fluctuations have also been found to be of lower hybrid as well as cyclotron types in our work as special cases in linear limits. This, in turn, can provide the nature of Alfven waves in plasmas in accordance with with their chaotic developments. Therefore, in particular, this result can potentially be very useful for dealing with chaotic developments of nonlinear Alfven waves in flowing MHD plasma systems. This has to be noted here that investigations on chaotic Alfven waves are scarcely reported till now except the work of Buti and Nocera [9] who has analysed chaotic Alfven waves in solar wind two decades ago.

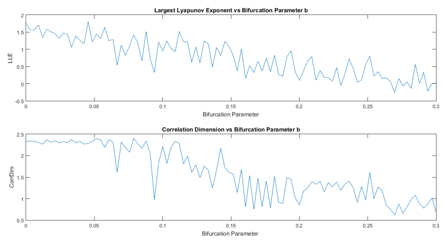

FIG. 2: Variations of LLE and CD for B_y against the parameter b keeping other parameters same as in FIG. 1

We know that fixed points in a system characterize stable and unstable equilibria in it according to their corresponding stability behaviour in phase space [10]. The fixed points in our system of Eqs. (1) and (2) are found to be

$$(B_y, \frac{dB_y}{dt}) = (\pm\sqrt{e^2 + \alpha {M_A}^2 - 1}, 0);\ (e, 0)\ and\ (B_z, \frac{dB_z}{dt}) = (\pm\sqrt{f^2 + \alpha {M_A}^2 - 1}, 0);\ (f, 0) \tag{3}$$

The detailed stability analysis of these fixed points has been performed in our work using Jacobian matrices [10]. We subsequently proceed to find numerical solutions for B_y and B_z using the fourth order Runge-Kutta (RK4) method in MATLAB. The typical numerical solutions pertaining to chaotic magnetic fields arising from the coupled nonlinear Eqs. (1) and (2) have been shown in FIG. 1 for arbitrary values of parameters along with plots corresponding to their phase space, FFT and power spectrum. The phase space plots have been shown in order to visualize the chaotic trajectories of nonlinear magnetic fields on phase plane. Similarly FFT diagrams have been shown explicitly to visualize B_y and B_z in frequency domain using discrete Fourier transform (DFT) [10]. The peaks in FFT diagrams characterize dominant modes of oscillations present in B_y and B_z with specific frequencies. Then the plots corresponding to power spectrum have been shown in order to explore the distributions of various frequencies of oscillations using $log - log$ graph. The slopes of these log-log graphs can be used to investigate various distributions such as power law for explaining phenomenon of self-organized criticality (SOC). A detailed study based on power spectrum is still in progress and planned to be reported in future. For simplicity, we now proceed to numerically analyse the dynamical features of B_y

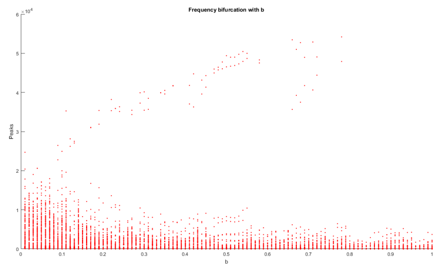

FIG. 3: Frequency bifurcations for B_y against the parameter b keeping other parameters same as in FIG. 1

only against the parameter b, which depends upon ν, α, Ω_e and Ω_i as specified earlier, keeping other parameters fixed. This physically implies that the dynamical features of B_y against variations of electron-ion collision frequency ν are to be explored in the present work as α, Ω_e and Ω_i are constants. A detailed study of the dynamical features of both B_y and B_z in the entire parameter space has been planned to be reported in future.

In the present work, the chaotic nature of B_y has been proved using Lyapunov exponent (LE) [10]. We know that the divergence of two infinitesimally closed trajectories in phase space can be quantified in terms of LE whose positive values represent chaotic behaviour [10]. We have shown the variations of the largest Lyapunov exponent (LLE) against the parameter b, which is also known as bifurcation parameter in our system to be explained later in this article, in FIG. 2 and these are found to be positive characterizing the chaotic nature of B_y in the considered parameter space. In addition, the variations of correlation dimension (CD) against variations in b have been shown in FIG. 2 explicitly; it has been seen that the values of correlation dimension initially lie between 2 and 2.5 which subsequently decreases to settle around 1 with slight fluctuations as b increases further. The fractal nature of oscillations in B_y can be visualized from these typical variations of CD. Its detailed analysis is in progress. We have also seen characteristic frequency bifurcations of fluctuations in B_y using peaks in FFT plots. Specifically we have analysed the variations of dominant peaks in FFT plots with respect to variations in b. These variations of peaks characterizing typical bifurcating nature can be visualized in FIG. 3 against the variations of the parameter b. In view of this fact, the parameter b has been referred to as bifurcation parameter in our work. Actually the dominant peaks in FFT plots represent dominant modes of oscillations having specific frequencies whose values can be determined directly from the plots themselves. The frequency bifurcations observed in fluctuations of B_y effectively imply that the dominant modes of oscillations in B_y with certain frequencies are to suffer qualitative changes with respect to variations in b which can be specified from careful observations of FIG. 3. We have already explored the fractal nature of fluctuations in B_y using correlation dimension as discussed earlier. Now we intend to investigate the behaviour of Hurst exponent H [12], which is also related to fractal dimension D as $D = 2 - H$ for self-similar time series, in order to explore long range dependences of oscillations in B_y. In particular, our aim is to explore the possibility of long range temporal correlations and anticorrelations in B_y. It is well-known that the values of H in the range $0 < H < 0.5$ represent long range anticorrelations whereas those in the range $0.5 < H < 1$ indicate the presence of long range correlations in the fluctuations; $H = 0.5$ can stand for a completely uncorrelated nature of fluctuations. It should be noted in this context that long range correlations basically imply the presence of long term positive autocorrelations; this represents the fact that a high value in the corresponding time series is probably to be followed by another high value and the values in the long time limit also tends to be high. Similarly long range anticorrelations indicate the signatures of long term negative autocorrelations having time series with long term switching between high and low values in adjacent pairs; this represents that a high value is to be probably followed by a low value and that the value after that will tend to be high, and this tendency to switch between high and low values continues for a long time in future. There are several methods for determination of H from a time series out of which rescaled range (R/S) analysis [13] is the oldest and best-known. In our work, R/S analysis has been employed to estimate Hurst exponent. The variations of H against variations in b has been shown in FIG. 4. From this figure, it is clear that the values of H always lie between 0.5 and 1 implying the presence of long range correlations in fluctuations of B_y considered for the specified parameter range. A detailed analysis of exploring the possibility of obtaining long range temporal correlations and anticorrelations in the entire parameter space is currently in progress. This is planned to be reported in future.

To conclude, we have analysed the chaotic evolution of nonlinear magnetic fields in flowing MHD plasmas by considering the governing coupled nonlinear equations derived by Lee and Parks [1]. Lower hybrid as well as cyclotron oscillations for magnetic field fluctuations have been reported in our work as special cases in linear limit. Then we proceed to find the fixed points along with their stability behaviour followed by numerical chaotic solutions. The typical trajectories of these chaotic solutions are also shown in phase space for a better understanding of their chaoticity in the context of LE. It has to be noted here that the exponential divergence of infinitesimally close trajectories in phase

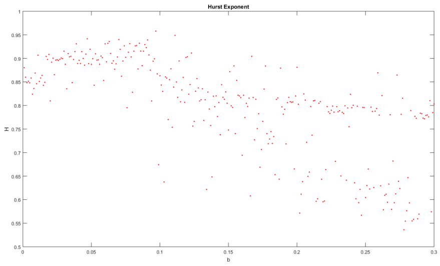

FIG. 4: Variations of Hurst exponents for B_y against variations of the parameter b keeping other parameters same as in FIG. 1

space is to be characterized by LE whose positive values indicate chaotic nature citeStrogatz. The chaotic nature of the numerical solutions has been proved by positive values of the largest Lyapunov exponent (LLE) in our work. The most important point in this context is that Lee and Parks [1, 2] have analysed the chaotic developments of nonlinear magnetic fields by reducing their governing equations to the forced Duffing equation. It is obvious that a forced second order equation can have chaotic solutions [10]. But, in the present work, we have reported the typical chaotic solutions of nonlinear magnetic fields even without a forcing term in the governing coupled nonlinear Eqs. (1) and (2) which hve not been explored in [1, 2]. Therefore, this happens to be a crucial result in our work which interprets the chaotic features of second order nonlinear equations even without any external forcing term. The chaotic oscillations of nonlinear magnetic fields have been analysed in frequency domain as well using FFT. In the FFT plots, the dominant peaks represent different modes of oscillations with different frequencies. These dominant peaks are observed to show bifurcating behaviour against the parameter b known as bifurcation parameter. This typical bifurcation is referred to as frequency bifurcation which signifies that the dominant modes of oscillations of chaotic magnetic fields with specific frequencies are to change qualitatively as the bifurcation parameter b changes as discussed earlier. A detailed analysis on distributions of these dominant modes of oscillations is in progress based on power spectrum with an intention to investigate certain phenomena like self-organized criticality (SOC) and is planned to be reported in future. We have also analysed the behaviour of correlation dimension of oscillations in B_y which shows their fractal nature. The correlation dimension is a type of fractal dimension. This further inspires us to explore the possibility of existence of long range correlations and anticorrelations in our system using Hurst exponent which is related to fractal dimension as well. Using Hurst exponent, we observe the existence of long range positive autocorrelations in fluctuations of B_y as discussed before. These special dynamical features in nonlinear magnetic field oscillations leading to chaotic developments in flowing MHD plasmas are desirably novel and have not been reported earlier as far as our knowledge goes. This kind of study can have significant implications in many laboratory, space and astrophysical plasmas which need to be investigated in MHD framework. Specifically the typical dynamical features of chaotic magnetic field fluctuations presented in this article can prove to be very useful predominantly in the contexts of plasma confinement and transport occurring through different dynamical features of charged particles in chaotic magnetic fields.

One of the authors Siba Prasad Acharya is pleased to acknowledge the financial assistance received from Department of Atomic Energy (DAE) under Government of India through institute fellowship scheme during this work. A. N. Sekar Iyengar would like to thank Scientific Engineering and Research Board (SERB) of Department of Science and Technology (DST) under Government of India for financial support in this work through project number CRG/2020/001747.

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